1. Exercises from Sections 1.4-1.7

Problem 1. *(Folland 1.5.8) If $S \subseteq \mathbb{R}^n$ is an infinite bounded set then S has an accumulation point*

Proof.
- Let $\{x_k\} \subseteq S$ be any sequence (it exists, since S is infinite)
- By theorem 1.19 since S is bounded there is a convergent subsequence $\{x_{k_n}\}$
- Let $a = \lim_{n\to\infty} x_{k_n}$; claim that a is an accumulation point of S.
- Let $\epsilon > 0$ be given and consider $B(\epsilon, a)$. Pick N large enough that $|x_{k_n} - a| < \epsilon$ for all $n > N$, which we can do since x_{k_n} converges to a. Then $x_{k_n} \in B(\epsilon, a)$ for all $n > N$
- This shows that every neighbourhood of a contains infinitely many points of S, therefore a is an accumulation point.

□

Problem 2. *(Folland 1.5.10) Let $\{x_k\}$ be a bounded sequence in $\mathbb{R}$. Define the following:*

$$\liminf_{k\to\infty} x_k = \lim_{n\to\infty} \inf\{x_k : n \geq k\}$$

$$\limsup_{k\to\infty} x_k = \lim_{n\to\infty} \sup\{x_k : n \geq k\}$$

Try and explain to them what liminf and limsup are by drawing them the picture on the wikipedia article.

Claim: There exists subsequences x_{k_n} and x_{k_m} such that $x_{k_n} \to \liminf x_k$ and $x_{k_m} \to \limsup x_k$.

Proof: We'll do the proof for $\liminf$ and the $\limsup$ case is identical.

- Set $x_{k_n} = \inf\{x_k : k \geq n\}$.
- Sequence is bounded because x_k was bounded to begin with
- Sequence is monotone by construction
- By the monotone convergence theorem (theorem 1.16) the limit exists and is unique
- By definition, $x_{k_n} \to \liminf_{k\to\infty} x_k$ so that is the limit.

Problem 3. *(Folland 1.5.12) Show that $\{x_k\}$ converges if and only if $\limsup x_k = \liminf x_k$, in which case the limit agrees with $\lim_{k\to\infty} x_k$*

Forward direction: If $\{x_k\}$ converges, then any subsequence converges to the same limit; $x_{k_n} = \inf_{n\geq k} x_k$ is such a subsequence whose limit converges to $\liminf x_k$ (and similarly for $\limsup$).

Backwards direction: We assume that $\liminf x_k = \limsup x_k$ and apply the sandwich theorem. For every k we have the following inequalities:

$$\inf_{n\geq k} x_n \leq x_k \leq \sup_{n\geq k} x_n \Longrightarrow \liminf x_k \leq \lim_{k\to\infty} x_k \leq \limsup x_k$$

So x_k converges.

Problem 4. *(Folland 1.6.2b) If $f : \mathbb{R}^n \to \mathbb{R}^m$ is everywhere continuous and $S \subseteq \mathbb{R}^n$ is bounded, then $f(S)$ is bounded*

- Notice that $S \subseteq \overline{S}$ implies that $f(S) \subseteq f(\overline{S})$ so it suffices to show that $f(\overline{S})$ is bounded
- S bounded $\Rightarrow$ $\overline{S}$ is bounded. There exists R such that $S \subseteq B(R, 0)$, then $\overline{S} \subseteq \overline{B}(R, 0)$, so fix some $\epsilon > 0$ then $\overline{S} \subseteq B(R + \epsilon, 0)$.
- Now $\overline{S}$ is closed and bounded, therefore compact by definition.
- f is now continuous on S, a compact set. The image $f(S)$ is compact, by theorem 1.22, therefore bounded.

Recall a set S is disconnected if there exists sets S_1 and S_2 with disjoint closures such that $S_1 \cup S_2 = S$. Otherwise, a set is called connected.

PROBLEM 5. *(Folland 1.7.1) Show the following sets are disconnected: (1) A finite set of more than two distinct points (2) A hyperbola $x^2 - y^2 = 1$ (3) $\{(x, y, z) \in \mathbb{R}^3 \,|\, xyz > 0\}$*

(1) Let $S = \{x_1, \ldots, x_n\}$, then set $S_1 = \{x_1\}$ and $S_2 = \{x_2, \ldots, x_n\}$. These are obviously disjoint and their unions are S.

(2) Let $S = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 - y^2 = 1\}$. The constraint implies $x = \pm\sqrt{1 + y^2}$, so set $S_1 = \{(x, y) \in \mathbb{R}^2 \,|\, x < 0, x^2 - y^2 = 1\}$ and $S_2 = \{(x, y) \in \mathbb{R}^2 \,|\, x > 0, x^2 - y^2 = 1\}$. (Draw a picture) This shows the set is disconnected.

(3) Let $S_1 = \{(x, y, z) \,|\, x > 0, y > 0, z > 0\}$ and $S_2 = \{(x, y, z) \,|\, x < 0, y < 0, z > 0\} \cup \{(x, y, z) \,|\, x < 0, y > 0, z < 0\} \cup \{(x, y, z) \,|\, x > 0, y < 0, z < 0\}$, then these sets are disjoint and their union gives S.

PROBLEM 6. *(Folland 1.7.7) S is disconnected if and only if there is a continuous function $f : S \to \mathbb{R}$ such that $f(S) = \{0, 1\}$*

Forward: Pick S_1 and S_2 with disjoint closures such that S is the union. Define $f : S \to \mathbb{R}$ by $f(S_1) = 0$ and $f(S_2) = 1$. We claim that our function is continuous. First, a lemma:

LEMMA 1.1. *If $K \subseteq \mathbb{R}^n$ is compact, $A \subseteq \mathbb{R}^n$ is closed, and $A \cap K = \emptyset$ then there exists $\delta > 0$ such that $|x - y| \geq \delta$ for all $x \in K$, $y \in A$.*

PROOF. Exercise for the reader. □

Let $x \in S$, then either $x \in S_1$ or $x \in S_2$. If we fix $\epsilon \geq 1$ then $f(S) \subseteq B(f(x), \epsilon)$; otherwise, fix $1 > \epsilon > 0$ and suppose without loss of generality that $x \in S_1$ (if $x \in S_2$, then an identical proof will hold). Since a point is compact, $\overline{S}_2$ is closed, and $x \notin \overline{S}_2$ by assumption of disconnectedness, by the lemma there exists $\delta > 0$ such that $|x - y| \geq \delta$ for all $y \in \overline{S}_2$. This means that for any $S \cap B(\delta, x) = S_1 \cap B(\delta, x)$, so we can always force $|f(x) - f(y)| = 0 < \epsilon$ so long as we pick $y \in B(\delta, x) \cap S$.

Backward: Suppose that we have a continuous function $f : S \to \mathbb{R}$ such that $f(S) \subseteq \{0, 1\}$, then set $S_1 = f^{-1}(0)$ and $S_2 = f^{-1}(1)$ which are disjoint closed sets whose union is S.